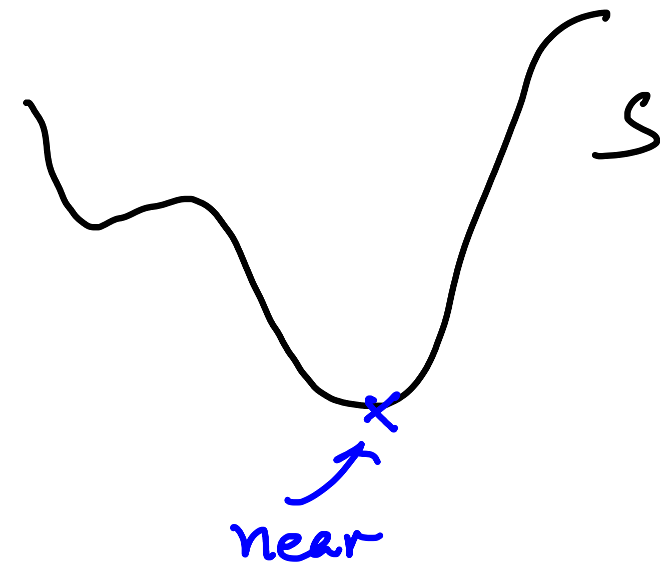
# §2. Perturbative theory and Feynman Diagram

**Recall**: $\int e^{-S/\hbar}$ = path integral"

**Today**: Asymptotic $\hbar$-expansion around a minimal of $S$




This has very nice combinatorial expression — Feynman diagrams, which also has physical interpretations.

We present this result via the BV idea.

**Last time**: Given volume form

$$\Omega = e^{f(x)} dx^1 \wedge \cdots \wedge dx^n$$

We can consider the integration map

$$\int : \mathcal{A} \longmapsto \mathbb{R}$$

certain functions on $\{x^i, \theta_i\}$ where $\theta_i$'s are anti commuting variables $\theta_i \theta_j = -\theta_j \theta_i$

If we integrate over $\mathbb{R}^n$, $\int_{\mathbb{R}^n}$ picks only the component without $\theta_i$'s (i.e. $PV^0$-part).

and
$$\int_{\mathbb{R}^n} : f(x) \longmapsto \int_{\mathbb{R}^n} f(x)\, \Omega$$

This integration is defined on $\Delta$-homology where $\Delta : \mathcal{A} \longmapsto \mathcal{A}$ is the BV-operator

$$\Delta = \sum_i \frac{\partial}{\partial x^i} \frac{\partial}{\partial \theta_i} + \sum_i (\partial_i f) \frac{\partial}{\partial \theta_i}$$

Eg: $$\int_{\mathbb{R}^n} \Delta(\varphi^i(x)\theta_i)\, \Omega = 0$$

Explicitly: $\int_{\mathbb{R}^n} \left( \sum_i \partial_i \varphi^i + \sum_i \varphi^i \partial_i f \right) e^f d^n x = 0$

this is just integration by part.

- Gaussian integral

Consider the simplest example $\mathbb{R}$ and

$$\Omega = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \qquad \text{Gaussian}$$

We study the integration map on polynomial func's

$$\int : \mathbb{R}[x] \longmapsto \mathbb{C}$$

$$g(x) \longmapsto \int_{\mathbb{R}} g(x)\, \Omega$$

or more generally

$$\int : \mathbb{R}[x, \theta] \longrightarrow \mathbb{C}$$

$$g(x) + h(x)\theta \longrightarrow \int_{\mathbb{R}} g(x)\, \Omega$$

The BV operator reads

$$\Delta = \frac{\partial}{\partial x}\frac{\partial}{\partial \theta} - x\frac{\partial}{\partial \theta} \qquad \left(f = -\tfrac{1}{2}x^2 \text{ here}\right)$$

Given any polynomial $g(x) \in \mathbb{R}[x]$, we have

$$\Delta g = 0 \qquad (\text{since } g \text{ has no } \theta\text{'s})$$

Let $[g]_\Delta$ denote the $\Delta$-coh classes:

$$[g_1]_\Delta = [g_2]_\Delta \Leftrightarrow g_1 - g_2 = \Delta \eta \text{ for some } \eta \in \mathbb{R}[x,\theta]$$

Then $\int$ is well-defined on $\Delta$-coh classes

$$\int g_1 \Omega = \int g_2 \Omega \quad \text{if } [g_1]_\Delta = [g_2]_\Delta$$

We also have the normalization

$$\int 1\, \Omega = 1 \quad (\text{by Gaussian integral})$$

Eg:
$$\Delta(x^{m-1}\theta) = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial\theta} - x\frac{\partial}{\partial\theta}\right)(x^{m-1}\theta)$$
$$= (m-1)x^{m-2} - x^m$$
$$\Rightarrow [x^m]_\Delta = (m-1)[x^{m-2}]_\Delta$$
$$\Rightarrow [x^{2k}]_\Delta = (2k-1)!!\,[1]$$
$$\Rightarrow \int_{\mathbb{R}} x^{2k}\,\Omega = (2k-1)!!\int_{\mathbb{R}}\Omega = (2k-1)!!$$
#

To organize the data, consider the following operator
$$\mathcal{U} = e^{\frac{1}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial x}} : \mathbb{R}[x,\theta] \longmapsto \mathbb{R}[x,\theta]$$

Explicitly, $\mathcal{U}(g(x)+h(x)\theta)$
$$= (\mathcal{U}g(x)) + (\mathcal{U}h(x))\theta$$
where $\mathcal{U}$ acts via Taylor expansion
$$\mathcal{U} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\right)^k$$

this is well-defined on $\mathbb{R}[x]$

<u>Lemma</u>: $\Delta = \mathcal{U}^{-1}\left(-x\frac{\partial}{\partial\theta}\right)\mathcal{U}$

i.e. $\Delta$ is conjugate to the simple operator $-x\frac{\partial}{\partial\theta}$ via the operator $\mathcal{U}$.

<u>Pf</u>: Exercise. #

As a result, we find a cochain isomorphism of complexes

$$\mathcal{U}: (\mathbb{R}[x,\theta], \Delta) \longmapsto \left(\mathbb{R}[x,\theta], -x\frac{\partial}{\partial\theta}\right)$$

$$1 \longmapsto 1$$

Cochain map means that

$$\mathcal{U}\circ\Delta(\varphi) = \left(-x\frac{\partial}{\partial\theta}\right)\circ\mathcal{U}(\varphi)$$

i.e. $\mathcal{U}$ intertwines $\Delta$ w/. $-x\frac{\partial}{\partial\theta}$

Observation: $H^\bullet\left(\mathbb{R}[x,\theta], -x\frac{\partial}{\partial\theta}\right) = H^0 = \mathbb{R}$

Let $[-]_{-x\partial_\theta}$ represent the $\left(-x\frac{\partial}{\partial\theta}\right)$-coh classes

Then for any $x^m = \left(-x\frac{\partial}{\partial\theta}\right)\left(-x^{m-1}\theta\right)$ if $m>0$.

$$\Longrightarrow \quad [h(x)]_{-x\partial_\theta} = [h(0)]_{-x\partial_\theta}$$

Now for any $g(x) \in \mathbb{R}[x]$

$$\begin{array}{ccc} [g(x)]_\Delta & \stackrel{\mathcal{U}}{\longmapsto} & [\mathcal{U}(g(x))]_{-x\partial_\theta} \\ \| & & \| \\ \mathcal{U}(g)(0)[1]_\Delta & \longleftarrow & [\mathcal{U}(g)(0)]_{-x\partial_\theta} \end{array}$$

We find $[g(x)]_\Delta = \mathcal{U}(g)(0)\,[1]_\Delta$

In other words,

$$\int_{\mathbb{R}} g(x)\,\Omega = e^{\frac{1}{2}\partial_x^2} g(x)\Big|_{x=0} \int_{\mathbb{R}} 1\,\Omega$$
$$= e^{\frac{1}{2}\partial_x^2} g(x)\Big|_{x=0}$$

In general, we can introduce a parameter $a$

**Prop:**
$$\boxed{\int_{\mathbb{R}} g(x+a)\,\Omega = e^{\frac{1}{2}\partial_a^2} g(a)} \quad \forall g \in \mathbb{R}[x]$$

**RK:** the shift by $a$ has the interpretation of effective fields as well as background fields later.

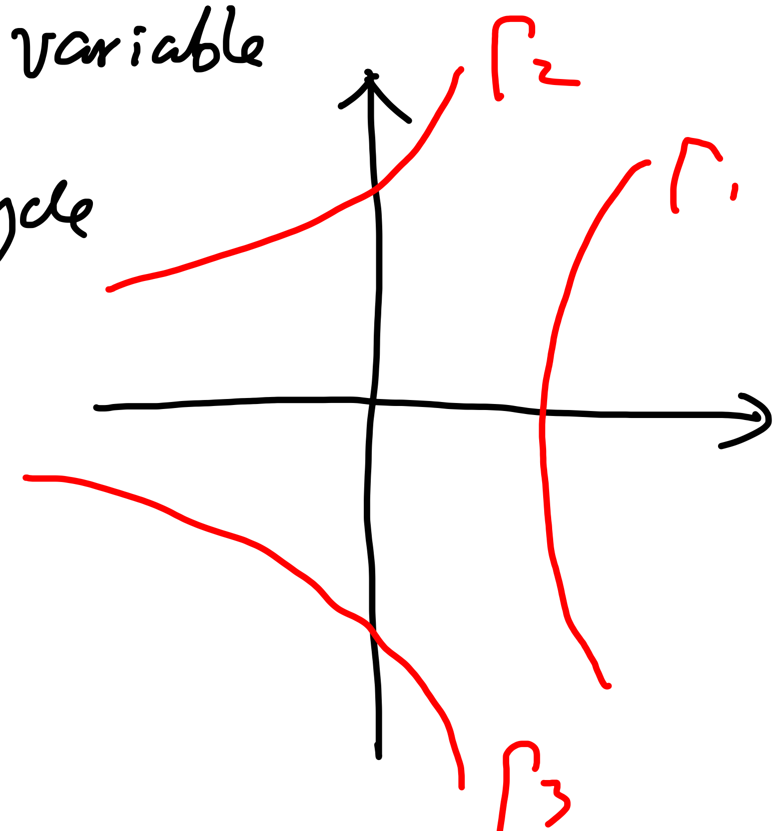
**Upshot:** the integration $\int(\ )\,\Omega$ is fully described by the operator $\mathcal{U}$.

Now we consider a Toy model:

$$\int_{\mathbb{R}} e^{\left(-\frac{1}{2}x^2+\frac{\lambda}{3!}x^3\right)/\hbar} \frac{dx}{\sqrt{2\pi\hbar}}$$

This integral is in fact divergent since $x^3$ blows up quickly at $\infty$. Two ways out

① treat $x$ as a complex variable and change the integration cycle




$$\int_{\Gamma_i} ( \cdots )$$

"Airy integral"

② treat the above integral as an asymptotic series in $\lambda$ via

$$e^{\frac{\lambda}{3!}x^3/\hbar} = \sum_{n\geq 0} \frac{\left(\frac{\lambda}{3!}x^3/\hbar\right)^n}{n!}$$

Let's do ② $\rightsquigarrow$ Perturbative theory.

Let's also add the background parameter $a$

$$" \int_{\mathbb{R}} e^{\left(-\frac{1}{2}x^2 + \frac{\lambda}{3!}(x+a)^3\right)/\hbar} \frac{dx}{\sqrt{2\pi\hbar}} "$$

$$:= \sum_{n\geq 0} \int_{\mathbb{R}} e^{-\frac{1}{2\hbar}x^2} \frac{\left(\frac{\lambda}{3!\hbar}(x+a)^3\right)^n}{n!} \frac{dx}{\sqrt{2\pi\hbar}}$$

Similarly, the integration $\int_{\mathbb{R}} (\quad) e^{-\frac{1}{2\hbar}x^2} \frac{dx}{\sqrt{2\pi\hbar}}$

is described by the operator $e^{\frac{1}{2}\hbar \partial_x^2}$

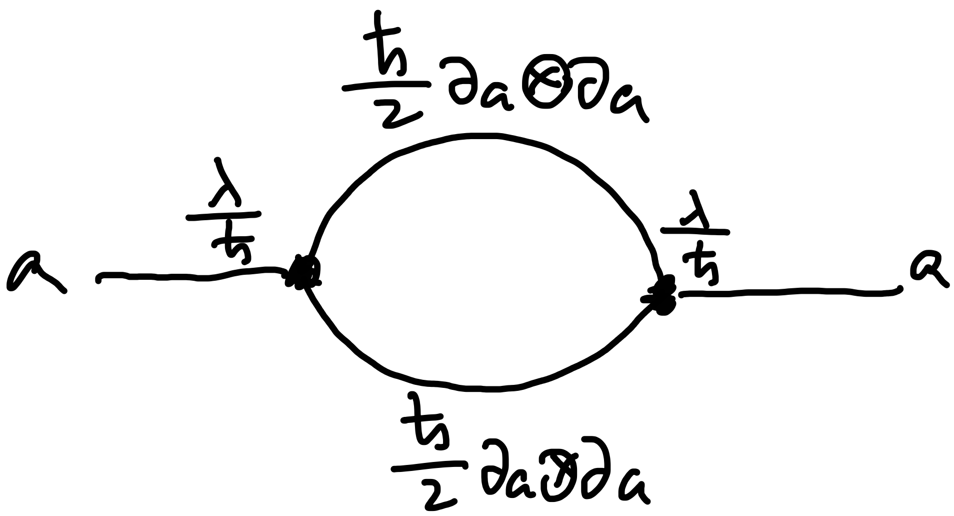
$$= e^{\frac{\hbar}{2}\partial_a^2} \quad e^{\frac{\lambda}{3!}a^3/\hbar}$$

$$= \sum_{k,m\geq 0} \frac{\left(\frac{\hbar}{2}\partial_a^2\right)^k}{k!} \frac{\left(\frac{\lambda}{3!\hbar}a^3\right)^m}{m!}$$

This infinite sum can be organized into graphs.

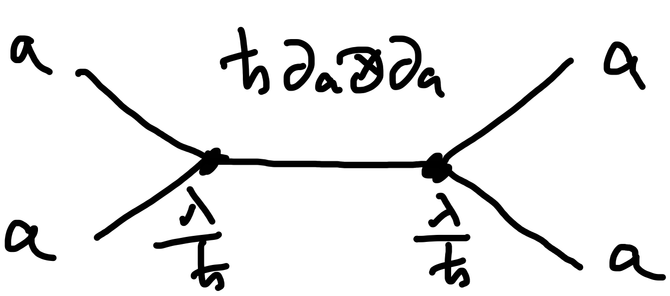
Here are some examples:

one term in $\left(\frac{\hbar}{2}\partial_a^2\right)^2\left(\frac{\lambda}{3!\,\hbar}a^3\right)^2$ has




$\sim \quad a^2$

one term in $\left(\frac{\hbar}{2}\partial_a^2\right)\left(\frac{\lambda}{3!\,\hbar}a^3\right)^2$ has




$\sim \quad \frac{1}{\hbar}a^4$

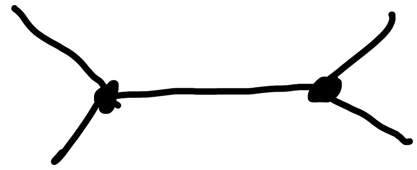
In general, given a graph $\Gamma$, let

$D$ = # of external edges

$E$ = # of internal edges

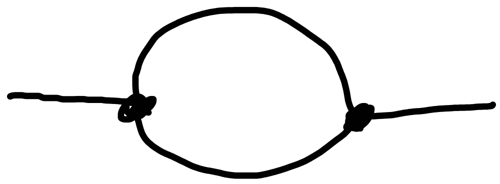
$V$ = # of vertices

e.g.



$D=4 \quad E=1 \quad V=2$

$\ell=0$



$D=2 \quad E=2 \quad V=2$

$\ell=1$

Define $w_\Gamma(a) = a^D \lambda^V \hbar^{E-V}$

For $\Gamma$ a connected graph, we have

$$V - E = \underbrace{\chi(\Gamma)}_{\text{Euler }\#} = 1 - \ell \qquad \ell : \# \text{ of loops}$$

$$\Rightarrow \boxed{W_\Gamma(a) = a^D \lambda^V \hbar^{\ell-1}}$$

**Prop**: $\displaystyle\int_{\mathbb{R}} e^{\left(-\frac{1}{2}x^2 + \frac{\lambda}{3!}(x+a)^3\right)/\hbar} \frac{dx}{\sqrt{2\pi\hbar}} := e^{\frac{\hbar}{2}\partial_a^2} e^{\frac{\lambda a^3}{3!\hbar}}$

$$= \exp\left(\sum_{\substack{\Gamma : \text{conn} \\ \text{trivalent graph}}} \frac{W_\Gamma(a)}{|\mathrm{Aut}\,\Gamma|}\right)$$

Here $\mathrm{Aut}(\Gamma)$ = automorphism groups of $\Gamma$

If we define

$$W(a) = \hbar \sum_{\Gamma:\text{conn}} \frac{W_\Gamma(a)}{|\mathrm{Aut}(\Gamma)|}$$

$$\underset{\text{in } \hbar}{\overset{\text{expand}}{=\!=}} \sum_{g\geq 0} W_g(a)\hbar^g$$

Here $\hbar W_\Gamma(a) \sim \hbar^{E-V+1} = \hbar^{\ell}$. Then

we can write the above formula as

$$e^{W(a)/\hbar} = \int_{\mathbb{R}} e^{-\frac{1}{2}x^2/\hbar} e^{I(x+a)/\hbar} \frac{dx}{\sqrt{2\pi\hbar}}$$

for $I(x) = \frac{\lambda x^3}{3!}$ cubic "interaction"

We can further write it as

$$\boxed{e^{W(a)/\hbar} = e^{\frac{\hbar}{2}\partial_a^2} e^{I(a)/\hbar}}$$

and $e^{\frac{\hbar}{2}\partial_a^2}$ plays the role of integration

Now we give this operator a name:

$$P := \frac{1}{2}\partial_x^2 \qquad = \text{"propagator"}$$

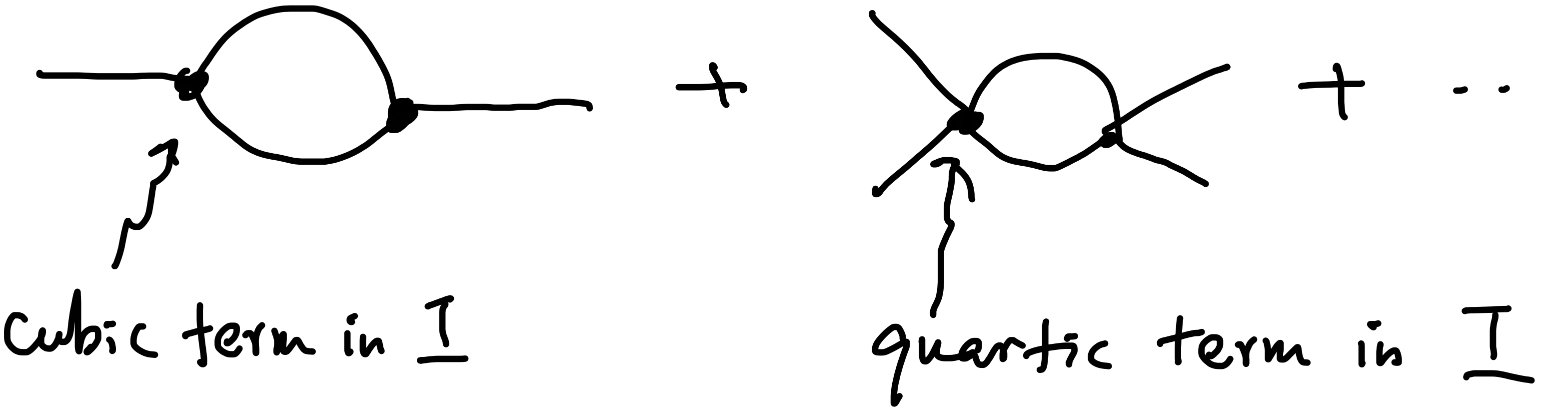
Define a transformation on $I(x)$

$I \longmapsto W(P,I)$ by the equation

$$e^{W(P,I)/\hbar} := e^{\hbar P} e^{I/\hbar}$$

Similarly, we have a graph formula

$$W(P,I) = \hbar \sum_{\Gamma:\,\mathrm{Conn}} \frac{W_\Gamma}{|\mathrm{Aut}(\Gamma)|}$$



**Prop**: $W(P,-)$ defines a transformation on

$$W(P,-): \mathbb{R}[[x,\hbar]]^+ \longmapsto \mathbb{R}[[x,\hbar]]^+$$

where $\mathbb{R}[[x,\hbar]]^+ := x^3\mathbb{R}[[x]] \oplus \hbar\mathbb{R}[[x,\hbar]]$

( "terms at least cubic modulo $\hbar$" )

$W(P,-)$: Renormalization group flow operator (RG) w.r.t. the propagator $P$.

**Ref Today**:

- K. Costello: "Renormalization and effective field theory"
  Ref for the renormalization group flow operator
- S. Li: "Intro to perturbative QFT and geometric applications"
  Note Part-I, Available at Homepage
- Bessis, Itzykson: "Quantum field theory techniques in graphical enumeration" Ref for diagram technique